Section 7 Finding an Inverse using Elementary Row Operations

The formula for the inverse of (3×3) and larger square matrices is much more complicated. We will see them in a later section. For now, we show a practical (but tedious) way to find the inverse of a matrix using “elementary row operations”. These operations are exactly the steps used in the “elimination” process for solving systems of equations.

The idea here is that the three elementary row operators:

Switching rows

Subtracting a constant times one row from another row

Multiplying an entire row by some number

can all be replicated by premultiplication by certain matrices. For example, to switch rows 1 and 2 in the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}$$

we can premultiply by

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is obtained by switching the 1st and 2nd rows of the (3×3) identity matrix. You can verify that premultiplying $\mathbf{A}$ by $\mathbf{E}$ switches rows 1 and 2 of $\mathbf{A}$:

$$\mathbf{EA} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 4 \\ 5 & 2 & 1 \end{bmatrix}$$

Similarly, premultiplying a matrix by

$$\mathbf{E}_{(2)\leftrightarrow(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which is obtained by switching the 2nd and 3rd rows of the (3×3) identity matrix, switches the first and third rows of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 4 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

This generalizes to switching other pairs of rows, and to larger matrices.

Another type of elementary row operation is to add/subtract a constant times one row to another row. For example, you can use the ‘3’ in row 1 as a pivot to eliminate the ‘5’ below it using this operation.

e.g. row2 = row2 - $\frac{5}{3}$row1 $\qquad \begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 2 & 4 \end{bmatrix}$

and use the resulting $\frac{1}{3}$ in row 2 as a pivot to eliminate the 2 in row 3:

e.g. row3 = row3 – $6 \times$row2 $\qquad \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 18 \end{bmatrix}$

These operations can also be done by premultiplying with appropriate matrices. To subtract $\frac{5}{3}$ of row 1 from row 2, premultiply the matrix by

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is obtained by taking an identity matrix and subtracting $\frac{5}{3}$ of row1 from row2. Verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 5 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 2 & 4 \end{bmatrix}$$

To subtract $6 \times$row 2 from row 3, premultiply by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

which is obtained by taking an identity matrix and subtracting $6 \times$row1 from row2. Verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 18 \end{bmatrix}$$

The pattern should be clear: to find the appropriate matrix for executing any particular row operation, take the identity matrix and apply that same operation to it.

To multiple row three by $\frac{1}{18}$, premultiply by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix}.$$

We have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{18} \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 1 \end{bmatrix}.$

If the inverse of a matrix $\mathbf{A}$ exists, then there is a series of elementary row operators that reduces $\mathbf{A}$ to the identity matrix. Suppose the required elementary row operators are (in order) $\mathbf{E}_1$, $\mathbf{E}_2$, ..., $\mathbf{E}_n$, then

$$\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}$$

which means that $\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 = \mathbf{A}^{-1}$. Furthermore, because $\mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{I} = \mathbf{E}_n \ldots \mathbf{E}_2 \mathbf{E}_1$, we can use the following technique:

Write $\mathbf{A}$ and $\mathbf{I}$ side-by-side. Then apply the same row operators to both $\mathbf{A}$ and $\mathbf{I}$ until $\mathbf{A}$ is reduced to the identity matrix. At the same time, the identity matrix will be "reduced" to the inverse matrix.

$$\begin{array}{r|l} \mathbf{A} & \mathbf{I} \\ \mathbf{E}_1\mathbf{A} & \mathbf{E}_1\mathbf{I} \\ \mathbf{E}_2\mathbf{E}_1\mathbf{A} & \mathbf{E}_2\mathbf{E}_1\mathbf{I} \\ \vdots & \\ \underbrace{\mathbf{E}_n \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A}}_{\text{reduced to } \mathbf{I}} & \underbrace{\mathbf{E}_n \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{I}}_{=\mathbf{A}^{-1}} \end{array}$$

Here is the fully worked out example for our example matrix:

$$\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 6 & 2 & 1 & 0 & 0 & 1 \end{array}$$

switch rows 1 and 2

$$\begin{array}{ccc|ccc} 3 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 6 & 2 & 1 & 0 & 0 & 1 \end{array}$$

row 3 minus $2 \times$ row 1,

$$\begin{array}{ccc|ccc} 3 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -2 & 1 \end{array}$$

$$\begin{array}{l} \text{row 2 minus } -\tfrac{4}{3}\times\text{row 3} \\ \text{row 1 minus } -\tfrac{2}{3}\times\text{row 3} \end{array} \qquad \left.\begin{array}{ccc} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{array}\right| \begin{array}{ccc} 0 & -\tfrac{1}{3} & \tfrac{2}{3} \\ 1 & -\tfrac{8}{3} & \tfrac{4}{3} \\ 0 & -2 & 1 \end{array}$$

$$\text{row 1 minus } \tfrac{1}{2}\times\text{row 2} \qquad \left.\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{array}\right| \begin{array}{ccc} -\tfrac{1}{2} & 1 & 0 \\ 1 & -\tfrac{8}{3} & \tfrac{4}{3} \\ 0 & -2 & 1 \end{array}$$

$$\begin{array}{l} \tfrac{1}{3}\times\text{row 1} \\ \tfrac{1}{2}\times\text{row 2} \\ -\tfrac{1}{3}\times\text{row 3} \end{array} \qquad \left.\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| \begin{array}{ccc} -\tfrac{1}{6} & \tfrac{1}{3} & 0 \\ \tfrac{1}{2} & -\tfrac{4}{3} & \tfrac{2}{3} \\ 0 & \tfrac{2}{3} & -\tfrac{1}{3} \end{array}$$

The inverse of $\mathbf{A} = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}$ is therefore

$$\begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\tfrac{1}{6} & \tfrac{1}{3} & 0 \\ \tfrac{1}{2} & -\tfrac{4}{3} & \tfrac{2}{3} \\ 0 & \tfrac{2}{3} & -\tfrac{1}{3} \end{bmatrix}.$$

You should verify this by computing

$$\begin{bmatrix} -\tfrac{1}{6} & \tfrac{1}{3} & 0 \\ \tfrac{1}{2} & -\tfrac{4}{3} & \tfrac{2}{3} \\ 0 & \tfrac{2}{3} & -\tfrac{1}{3} \end{bmatrix}\begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 2 \\ 6 & 2 & 1 \end{bmatrix}\begin{bmatrix} -\tfrac{1}{6} & \tfrac{1}{3} & 0 \\ \tfrac{1}{2} & -\tfrac{4}{3} & \tfrac{2}{3} \\ 0 & \tfrac{2}{3} & -\tfrac{1}{3} \end{bmatrix}$$

and checking to see if you get the identity matrix.

If we are using this technique to solve the equation $\mathbf{Ax} = \mathbf{b}$, i.e. to compute $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$: write down $\mathbf{A}$ and $\mathbf{b}$ side-by-side and apply the row operations to redure $\mathbf{A}$ into the identity matrix. The rationale for this is the same as previously:

$$\begin{array}{r|l} \mathbf{A} & \mathbf{b} \\ \mathbf{E}_1\mathbf{A} & \mathbf{E}_1\mathbf{b} \\ \mathbf{E}_2\mathbf{E}_1\mathbf{A} & \mathbf{E}_2\mathbf{E}_1\mathbf{b} \\ & \vdots \\ \underbrace{\mathbf{E}_n\ldots\mathbf{E}_2\mathbf{E}_1\mathbf{A}}_{\text{reduced to } \mathbf{I}} & \underbrace{\mathbf{E}_n\ldots\mathbf{E}_2\mathbf{E}_1\mathbf{b}}_{=\mathbf{A}^{-1}\mathbf{b}} \end{array}$$

Exercises

1. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{bmatrix}$.

2. The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -3 \\ 1 & 2 & -6 \end{bmatrix}$ has no inverse. Apply elementary row operations to $\mathbf{A}$ as though finding the inverse. What happens?

3. Write down the simultaneous equations

$$\begin{aligned} 3x_1 + 3x_2 + 2x_3 &= 6 \\ 2x_1 + x_2 + 3x_3 &= 3 \\ x_1 + 5x_3 + 2x_3 &= 4 \end{aligned}$$

in the form $\mathbf{Ax} = \mathbf{b}$. Solve this by

(i) finding the inverse of $\mathbf{A}$ and computing $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$;

(ii) writing down $\mathbf{A}$ and $\mathbf{b}$ side-by-side:

$$\begin{array}{ccc|c} 3 & 3 & 2 & 6 \\ 2 & 1 & 3 & 3 \\ 1 & 5 & 2 & 4 \end{array}$$

and applying the necessary row operations to reduce the left side of the matrix to the identity matrix.

4. Find the inverse of the matrix

$$\mathbf{D} = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 1 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{bmatrix}.$$